

Introduction to mixed-effects models

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Outline

- ① Mixed-effects models
- ② Longitudinal data analysis

① Mixed-effects models

Mixed-effects models

- Mixed-effects models are a class of statistical models that include fixed effects as well as random effects
- Fixed effects vs. random effects¹
 - For fixed effects, only effects of the factor levels used in the present study are considered (manipulated conditions, e. g., assigned groups, but also sex, or other variables ...)
 - Of interest is how these levels differ
 - For random effects, the factor levels considered in a study are regarded as a (random) sample from some population (e. g., words, raters, subjects, ...)
 - Of interest are conclusions about the underlying population and its variation

¹Some critical discussion on these definitions:

https://statmodeling.stat.columbia.edu/2005/01/25/why_i_dont_use/

Linear mixed-effects model

- The linear mixed-effects model has the general form

$$y_i = X_i \beta + Z_i v_i + \varepsilon_i$$

with fixed effects β , random effects v_i , and the design matrices X_i and Z_i and the assumptions

$$v_i \stackrel{iid}{\sim} N(0, \Sigma_v), \quad \varepsilon_i \stackrel{iid}{\sim} N(0, \sigma^2 I_{n_i})$$

- This implies for the marginal covariance matrix

$$\text{Cov}(y_i) = \Sigma_i = Z_i \Sigma_v Z_i' + \sigma^2 I_{n_i}$$

Linear mixed-effects model

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1p} \\ 1 & x_{21} & x_{22} & \dots & x_{2p} \\ 1 & x_{31} & x_{32} & \dots & x_{3p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & x_{N2} & \dots & x_{Np} \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} z_{10} & z_{11} & \dots & z_{1q} & \dots \\ z_{20} & z_{21} & \dots & z_{2q} & \dots \\ z_{30} & z_{31} & \dots & z_{3q} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{N0} & z_{N1} & \dots & z_{Nq} & \dots \end{pmatrix} \cdot \begin{pmatrix} v_{10} \\ \vdots \\ v_{1q} \\ v_{20} \\ \vdots \\ v_{Nq} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_N \end{pmatrix}$$

② Longitudinal data analysis

Longitudinal data

- Consist of repeated measurements on the same subject taken over time
- Are a frequent use case for mixed-effects models
- Contain time as a predictor: time trends within and between subjects are of interest

```
library(lme4)
data(sleepstudy)
?sleepstudy
str(sleepstudy)
summary(sleepstudy)
head(sleepstudy)
```

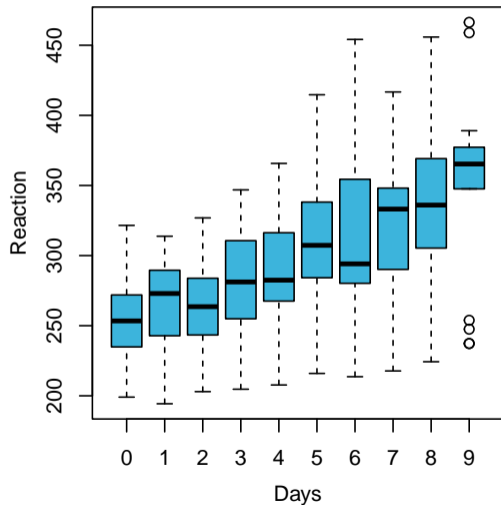
Sleep study

- Average reaction time per day for subjects in a sleep deprivation study
- On day 0, the subjects had their normal amount of sleep
- Starting that night they were restricted to 3 hours of sleep per night
- Observations represent the average reaction time on a series of tests given each day to each subject

A data frame with 180 observations on the following 3 variables

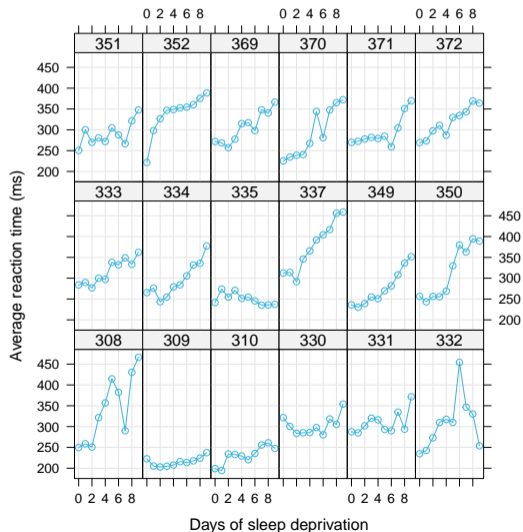
Reaction	Average reaction time (ms)
Days	Number of days of sleep deprivation
Subject	Subject number on which the observation was made

Visualization of data



```
boxplot(Reaction ~ Days, sleepstudy)
```

Visualization of individual data



```
library(lattice)
```

```
xyplot(Reaction ~ Days | Subject,
       data = sleepstudy,
       type = c("g", "b"),
       xlab = "Days of sleep deprivation",
       ylab = "Average reaction time (ms)",
       aspect = "xy")
```

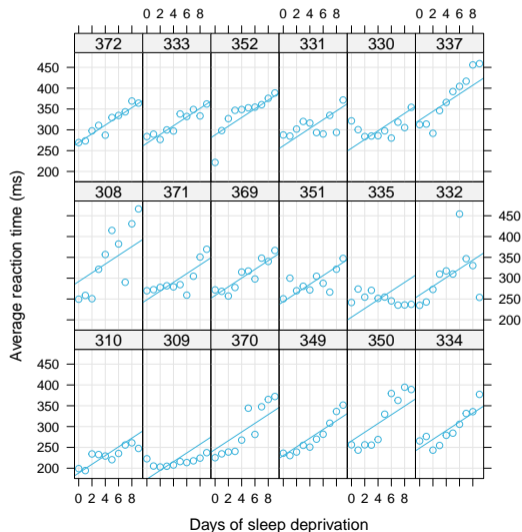
Random intercept model

- The random intercept model adds a random intercept for each subject

$$y_{ij} = \beta_0 + \beta_1 \text{Days}_{ij} + v_{0i} + \varepsilon_i$$

with $v_{0i} \stackrel{iid}{\sim} N(0, \sigma_v^2)$, $\varepsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma_\varepsilon^2)$

- The slope is identical for each subject (and the population)



Fixed and random effects for random intercept model

Subject	Days	Reaction	Fixed		Random	
			Intercept	Slope	SubInt	Resid
308	0	250	251.4	10.5	40.8	-42.6
308	1	259	251.4	10.5	40.8	-44.0
308	2	251	251.4	10.5	40.8	-62.3
⋮	⋮	⋮	⋮	⋮	⋮	⋮
308	8	431	251.4	10.5	40.8	54.7
308	9	466	251.4	10.5	40.8	80.0
309	0	223	251.4	10.5	-77.8	49.2
309	1	205	251.4	10.5	-77.8	21.2
309	2	203	251.4	10.5	-77.8	8.5
⋮	⋮	⋮	⋮	⋮	⋮	⋮
310	0	199	251.4	10.5	-63.1	10.8
310	1	194	251.4	10.5	-63.1	-4.4
310	2	234	251.4	10.5	-63.1	25.1
⋮	⋮	⋮	⋮	⋮	⋮	⋮
					σ_u^2	σ_ϵ^2

Random slope model

- The random slope model adds a random intercept and a random slope for each subject

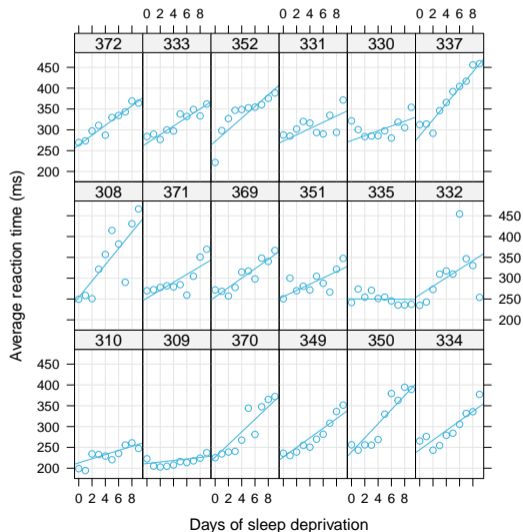
$$y_{ij} = \beta_0 + \beta_1 \text{Days}_{ij} + v_{0i} + v_{1i} \text{Days}_{ij} + \varepsilon_{ij}$$

with

$$\begin{pmatrix} v_{0i} \\ v_{1i} \end{pmatrix} \stackrel{iid}{\sim} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_v = \begin{pmatrix} \sigma_{v_0}^2 & \sigma_{v_0 v_1} \\ \sigma_{v_0 v_1} & \sigma_{v_1}^2 \end{pmatrix} \right)$$

$$\varepsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

- Individual slopes for each subject



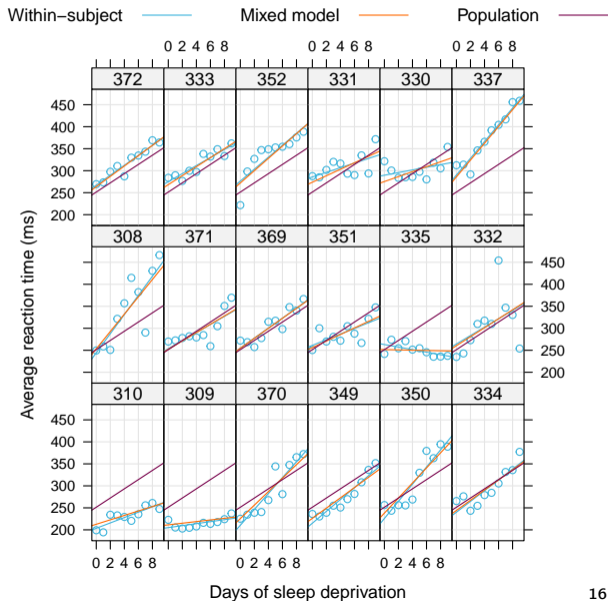
Fixed and random effects for random slope model

Subject	Days	Reaction	Fixed		Random		
			Intercept	Slope	SubInt	SubSlp	Resid
308	0	250	251.4	10.5	2.3	9.2	-4.1
308	1	259	251.4	10.5	2.3	9.2	-14.6
308	2	251	251.4	10.5	2.3	9.2	-42.2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
308	8	431	251.4	10.5	2.3	9.2	19.6
308	9	466	251.4	10.5	2.3	9.2	35.7
309	0	223	251.4	10.5	-40.4	-8.6	11.7
309	1	205	251.4	10.5	-40.4	-8.6	-7.6
309	2	203	251.4	10.5	-40.4	-8.6	-11.7
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
310	0	199	251.4	10.5	-39.0	-5.4	-13.4
310	1	194	251.4	10.5	-39.0	-5.4	-23.1
310	2	234	251.4	10.5	-39.0	-5.4	11.8
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$\sigma_{v_0}^2$ $\sigma_{v_0v_1}$ $\sigma_{v_1}^2$ σ_{ϵ}^2

Partial pooling

- Within-subject regression line shows regression line fitted to data for each individual
- Population regression line shows fixed effects for mixed-effects model
- Mixed model regression line shows individual regression lines as predicted by mixed-effects models



References

- Bates, D. (2010). *lme4: Mixed-effects modeling with R (book draft)*. Retrieved from <https://lme4.r-forge.r-project.org/>
- Bates, D., Mächler, M., Bolker, B., & Walker, S. (2015). Fitting linear mixed-effects models using lme4. *Journal of Statistical Software*, 67(1), 1–48. Retrieved from <https://CRAN.R-project.org/package=lme4/vignettes/lmer.pdf> doi: 10.18637/jss.v067.i01