The discrepancy between items and knowledge states

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Layer discrepancy by Doble et al. (2019)

Let $\mathcal{K}$ be a learning space on a domain Q and $q \in Q$ an item. Then the *first outer layer* and the *first inner layer* of a subset S of Q are defined as follows

$$S^{ol_1} := \{q \notin S \mid \forall p \notin S, p \preccurlyeq q \implies p = q\},$$
$$S^{il_1} := \{q \in S \mid \forall p \in S, q \preccurlyeq p \implies p = q\}.$$

The *n-th outer and inner layer* are for $n \geq 2$ recursively defined as

$$S^{ol_n} := (S \cup \bigcup_{i=1}^{n-1} S^{ol_i})^{ol_1},$$
$$S^{il_n} := (S \setminus \bigcup_{i=1}^{n-1} S^{il_i})^{il_1}.$$

The *outer layer discrepancy* is

$$d^o_{layer} : Q \times \mathcal{K}_q \to \mathbb{N},\ (q, K) \mapsto d^o_{layer}(q, K) := n \text{ for } q \in K^{ol_n}.$$

The *inner layer discrepancy* is

$$d^i_{layer} : Q \times \mathcal{K}_q \to \mathbb{N},\ (q, K) \mapsto d^i_{layer}(q, K) := n \text{ for } q \in K^{il_n}.$$

So the *layer discrepancy* is defined as follows:

$$d_{layer} : Q \times \mathcal{K} \to \mathbb{N},\ (q, K) \mapsto d_{layer}(q, K) := \begin{cases} d^o_{layer}(q, K) \text{ if } q \notin K, \\ d^i_{layer}(q, K) \text{ if } q \in K. \end{cases}$$

Minimum discrepancy

Let $\mathcal{K}$ be a knowledge structure defined on a domain Q, $q \in Q$ and $K \in \mathcal{K}$. Then we define:

$$K_{q,\supseteq} := \{L \in \mathcal{K}_q \mid K \supseteq L\},$$
$$K_{q,\subseteq} := \{L \in \mathcal{K}_q \mid K \subseteq L\}.$$

The *outer minimum discrepancy* is

$$d^o_{min} : Q \times \mathcal{K}_q \to \mathbb{N},\ (q, K) \mapsto d^o_{min}(q, K) := \min_{L \in K_{q,\subseteq}} \{|L \setminus K|\} \quad \text{for } q \notin K.$$

The *inner minimum discrepancy* is

$$d^i_{min} : Q \times \mathcal{K}_q \to \mathbb{N},\ (q, K) \mapsto d^i_{min}(q, K) := \min_{L \in K_{q,\supseteq}} \{|K \setminus L|\} \quad \text{for } q \in K.$$

So overall the *minimum discrepancy* is defined as follows:

$$d_{min} : Q \times \mathcal{K} \to \mathbb{N},\ (q, K) \mapsto d_{min}(q, K) := \begin{cases} d^o_{min}(q, K) \text{ if } q \notin K, \\ d^i_{min}(q, K) \text{ if } q \in K. \end{cases}$$

Lemma

Let $\mathcal{K}$ be a quasi ordinal knowledge space defined on a domain Q and $\preccurlyeq \subset Q \times Q$ the corresponding precedence relation. Then the following holds for all items $q \in Q$:

$$d^o_{min}(q, K) = |\{p \notin K \mid p \preccurlyeq q\}|,\ \forall K \in \mathcal{K}_q,$$
$$d^i_{min}(q, K) = |\{p \in K \mid q \preccurlyeq p\}|,\ \forall K \in \mathcal{K}_q.$$

Comparison of the discrepancies

Generalization

The layer discrepancy can be defined on arbitrary discriminative knowledge structures. Since equally informative items cannot be contained in any inner or outer layer, a generalization to non discriminative knowledge structures is not possible.

Observed Properties

- Minimum discrepancy depends on the knowledge structure.
- Layer discrepancy depends on the precedence relation only.

A fair comparison between layer and minimum discrepancy is possible for quasi ordinal learning spaces only. In quasi ordinal learning spaces...

- ...minimum discrepancy and layer discrepancy differ in distinct cases.
- ...minimum discrepancy represents the length of a learning path
- ...the first inner and outer layer equal inner and outer fringe respectively.

Disjoint union of chains

Let $\mathcal{K}$ be a quasi ordinal learning space on a domain Q and $\preccurlyeq \subseteq Q \times Q$ the corresponding precedence relation. The structure $\mathcal{K}$ satisfies the *duc - condition*, if for all $p, p', q \in Q$:

$$(p \preccurlyeq q \text{ and } p' \preccurlyeq q) \text{ or } (q \preccurlyeq p \text{ and } q \preccurlyeq p')$$
$$\implies p \preccurlyeq p' \text{ or } p' \preccurlyeq p$$

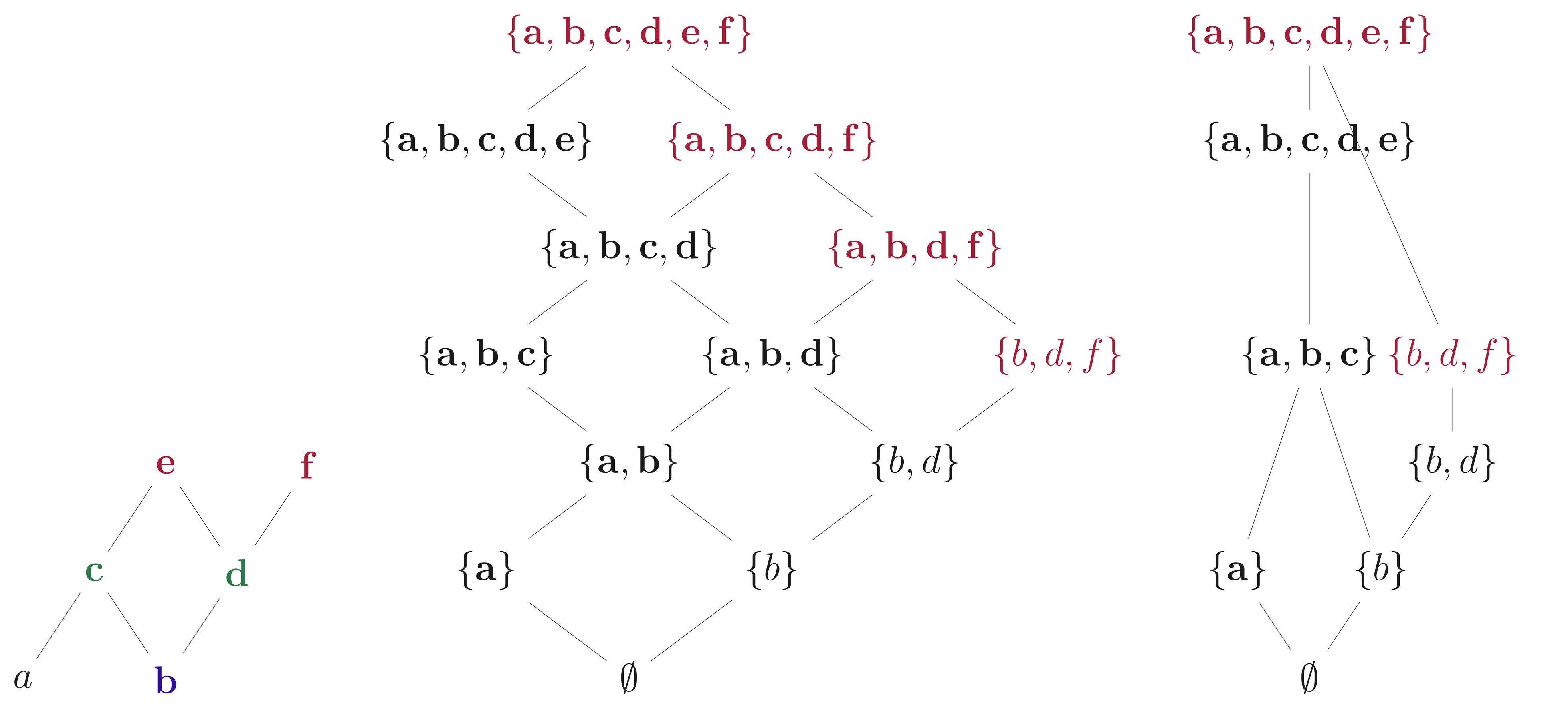
Theorem

Then the following equivalence holds:

$$\mathcal{K} \text{ satisfies the duc - condition} \iff$$
$$d_{layer}(q, K) = d_{min}(q, K),\ \forall q \in Q \text{ and } K \in \mathcal{K}$$

- The Hasse diagrams of precedence relations, which follow the duc-condition, are disjoint unions of chains.
- The duc-condition is a rather strict constraint.
- A partial order as the precedence relation or a quasi ordinal learning space which is a distributive graded lattice are not sufficient for duc.

Examples



(a) Precedence relation (b) Associated quasi ordinal knowledge space $\mathcal{K}_1$ (c) Associated smallest knowledge structure $\mathcal{K}_2$

Figure 2: Hasse diagramms for an example on a domain with 6 items.

Table 1: Minimum discrepancy and layer discrepancy for the example in Figure 1. The rows indicate states, columns indicate items. Negative numbers represent inner discrepancies, positive numbers outer discrepancies.

	$\mathcal{K}_1$												$\mathcal{K}_2$											
	d_{layer}						d_{min}						d_{layer}						d_{min}					
	a	b	c	d	e	f	a	b	c	d	e	f	a	b	c	d	e	f	a	b	c	d	e	f
$\emptyset$	-1	-1	**-2**	-2	**-3**	-3	-1	-1	**-3**	-2	**-5**	-3	-1	-1	-2	-2	-3	-3	-1	-1	-3	-2	-5	-3
$\{a\}$	1	-1	-2	-2	**-3**	-3	1	-1	-2	-2	**-4**	-3	1	-1	-2	-2	-3	-3	1	-2	-2	-4	-4	-5
$\{b\}$	-1	1	-2	-1	**-3**	-2	-1	1	-2	-1	**-4**	-2	-1	1	-2	-1	-3	-2	-2	1	-2	-1	-4	-2
$\{a, b\}$	1	1	-1	-1	**-2**	-2	1	1	-1	-1	**-3**	-2												
$\{b, d\}$	-1	2	-2	1	-3	-1	-1	2	-2	1	-3	-1	-1	2	-2	1	-3	-1	-3	2	-3	1	-3	-1
$\{a, b, c\}$	2	2	1	-1	-2	-2	2	2	1	-1	-2	-2	2	2	1	-1	-2	-2	2	2	2	-2	-2	-3
$\{a, b, d\}$	1	2	-1	1	-2	-1	1	2	-1	1	-2	-1												
$\{b, d, f\}$	-1	3	-2	2	-3	1	-1	3	-2	2	-3	1	-1	3	-2	2	-3	1	-3	3	-3	2	-3	1
$\{a, b, c, d\}$	2	**2**	1	1	-1	-1	2	**3**	1	1	-1	-1												
$\{a, b, d, f\}$	1	3	-1	2	-2	1	1	3	-1	2	-2	1												
$\{a, b, c, d, e\}$	3	**3**	2	2	1	-1	3	**4**	2	2	1	-1	3	3	2	2	1	-1	3	4	3	2	2	-1
$\{a, b, c, d, f\}$	2	**3**	1	2	-1	1	2	**4**	1	2	-1	1												
$\{a, b, c, d, e, f\}$	3	**3**	2	**2**	1	1	3	**5**	2	**3**	1	1	3	3	2	2	1	1	3	5	3	3	3	1

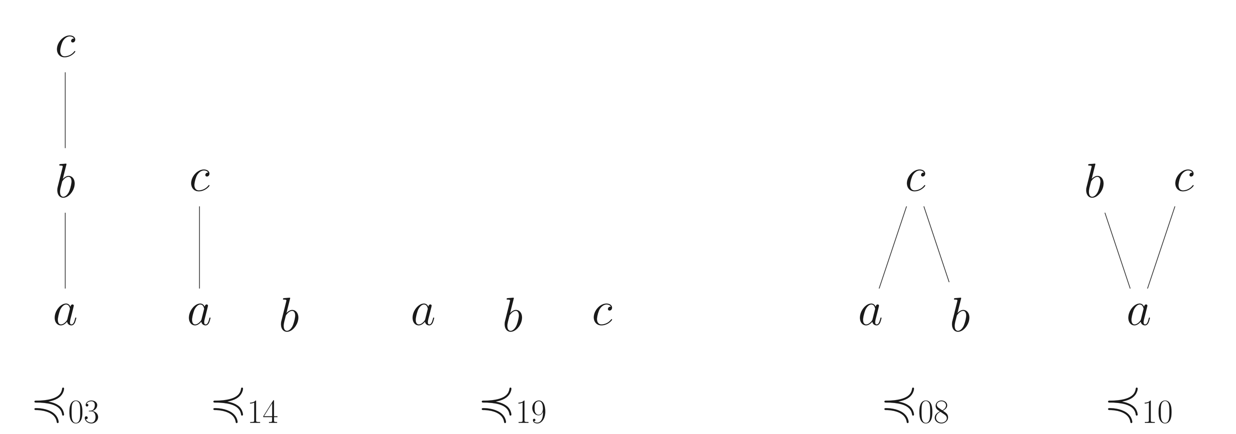


Figure 3: Hasse diagrams of all antisymmetric precedence relations on $Q = \{a, b, c\}$.

Motivation and future research

Same error probabilities for everyone?!

Item 1: Explain the hairy ball theorem!

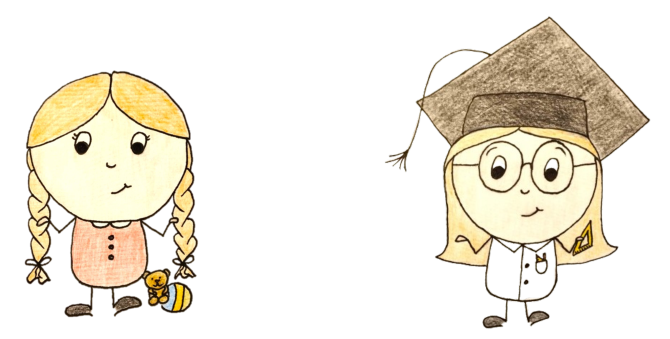


Figure 1: On the left an unexperienced student, who is new to the topic. On the right an experienced student, who knows all the prerequisites of Item 1.

Generalization of the BLIM:

- The assumption of constant error probabilities across persons should be relaxed.
- The response error probabilities β and η depend on the item and on the discrepancy between the item and the persons knowledge state.
- There should be as few new parameters introduced as possible.

References

Doble, C., Matayoshi, J., Cosyn, E., Uzun, H., and Karami, A. (2019). A data-based simulation study of reliability for an adaptive assessment based on knowledge space theory. *International Journal of Artificial Intelligence in Education*, 29(2).